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Fixed scale transformation for Ising and Potts clusters

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Abstract. The fractal dimension of Ising and Potts clusters are determined via the fixed scale transformation approach, which epxloits both the self-similarity and the dynamical invariance of these systems at criticality. The results are easily extended to droplets. A discussion of the interrelationships between the present approach and renormalization group methods as well as Glauber-type dynamics is provided.

1. Introduction

The fixed scale transformation, (FST), a novel technique [1, 2] introduced for computing the fractal dimension of Laplacian growth clusters, is applied here to the equilibrium problem of Ising clusters at criticality, in two dimensions. The method yields very good quantitative agreement with the known exact results. The same method has also recently been applied to the problem of percolation clusters and to invasion percolation, also with very good results [3, 4].

The fractal dimension D of the Ising clusters, i.e. the connected clusters of sites with identical spins, has been a controversial issue for a long time [5, 6 and references therein]. Recently it was shown [5, 7] that the exponents describing Ising clusters at the critical point are those of the dilute $q=1$ Potts model at its tricritical point, with the exact value $D=\frac{187}{96}=1.947$ in good agreement with numerical estimates.

Here we will undertake a direct evaluation of the fractal dimension of Ising clusters exploiting the scale invariance of the critical fluctuations and making explicit use of the statistical self-similarity under translation. It is an easy generalization to treat the q-state Potts model, with $2 \leqslant q \leqslant 4$ with our method, and we make explicit predictions for the fractal dimension of the q-state Potts clusters at criticality. To our knowledge, no exact results are available for $q=3, 4$ but our predictions are very close, numerically, to $D=2-x_{\mathrm{H}}$, where x_{H} is the magnetic exponent at the tricritical point of the dilute $q=2$ and $q=3$ Potts models. We also show that the Ising and Potts 'droplets' [6, 8, 9] can be defined in a very natural way in this approach, and we obtain reasonable estimates of the fractal dimensions. We also show that the Ising and Potts 'droplets' [6, 8, 9] can be defined in a very natural way in this approach, and we estimate their fractal dimension, obtaining reasonable, though slower convergence to the exact result.

The fixed scale transformation approach affords a new and accurate method for the determination of critical exponents. The FST was first developed to treat fractal growth models, but the basic ideas continue to apply here. Fractal growth models are

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'intrinsically critical', in the sense that they give rise to self-similar structures without the need to tune a parameter to its critical value, in contrast to equilibrium statistical models, such as the Ising model, where scale invariance of the fluctuations arises only at a special value of the couplings, namely, the critical point. Although in many instances the value of the critical couplings is known exactly from, say, duality relations, it is much more difficult to obtain reliable values for the critical exponents, or, the anomalous dimensions of the critical fluctuations. The FST provides a bridge between such non-universal quantities as the strength of the critical couplings or a particular lattice topology on the one hand and universal anomalous dimensions, on the other.

The FST method in the present context may be summarized as follows. The critical clusters are modelled by an appropriate ensemble of random Cantor sets, the ensemble being characterized by a scale invariant conditional correlation. At equilibrium, the conditional correlation functions at a generic length scale are determined from the fixed point of the fixed scale transformation, evaluated at the critical value of the couplings. Invariance under the FST corresponds to the requirement of translational invariance. The requirement that the equilibrium correlations be translationally invariant, as well as self-similar under scale change, should be compared with the similar requirement that an irreversible aggregation model be invariant under further growth [1, 2].

In the present scheme, the propagation of correlations between points of the cluster is treated as a sum over products of short-range correlations over different paths. It should be pointed out (also see section 5) that this is a convergent scheme which can be improved order by order in the length of the paths, or the size of the clusters, contributing to the relevant correlations. In the case of irreversible aggregation (or percolation considered as a growth process), convergence is established [1, 2, 3] in the order of the aggregation process, i.e. the number of particles added.

The paper is organized as follows. In section 2, the FST is applied to the Ising model at criticality and the fractal dimension of the Ising clusters is computed. In section 3, this is generalized to the Potts model. In section 4, the related problem of Ising and Potts droplets is treated. In section 5, we provide a discussion of certain common features between the renormalization group and the present method and also demonstrate its close relationship to a dynamical approach.

2. The fixed scale transformation applied to Ising clusters

Consider a segment of the intersection of the Ising cluster with a strip at any length scale 2^{-m}. The strip will consist of empty or occupied boxes, nesting inside one another in a *self-similar* manner. In particular, there will be an invariant probability C_1 (C_2) that under fine-graining by 2, a box belonging to the intersection set will reveal one (two) occupied box(es) at the next level (figure 1). These fine-graining configurations will be called type 1 and 2 respectively. The self-similarity dimension of the intersection set is now clearly given by $\ln(C_1+2C_2)/\ln 2$ with $C_1+C_2=1$. For a detailed discussion of this construction see [2].

Now consider a translation of this strip by one 'lattice constant' (at the given length scale) parallel to the original one. The distribution C_1' (C_2') should be the same on the intersection set thus obtained as on the previous one. We will proceed to build a transformation matrix that connects C_i' to C_i and look for its fixed point. To do this, we will interpret 'growth' in this context as the propagation of correlations at criticality.

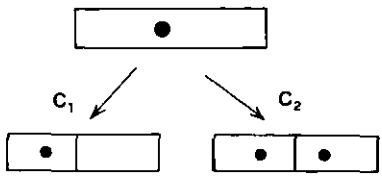

Figure 1. Under fine graining by a factor of 2, a cell belonging to the interaction set reveals configurations of type 1 or 2, with probabilities C_1, C_2.

It is important to note that the fractal object in which we are interested is described in terms of a new set of lattice gas variables whose statistics are not the same as those of the original spin system, although determined by it in a way we will discuss later. Since we are considering the system at a generic length scale, an 'occupied site' is a region at that particular length scale containing a connected part of the cluster of up-spins. Conversely, an 'empty site' may contain any number of up-spins which, however are bounded away from the connected cluster by a number of down-spins. Thus in this coarse-grained picture, the absence of a bond will signal an 'empty' site at the other end and *not* a down-spin, which is important for the correct weighting of the ensuing configurations.

That the statistics remain *invariant under scale change* follows from the fact that the underlying spin system exhibits self-similarity at the critical temperature where the correlation length diverges. Strictly speaking, for an Ising-like system with short-range interactions, a given coarse-graining procedure generates a flow in Hamiltonian space whose (critical) fixed point is at a value of the nearest neighbour (NN) interactions different from the 'bare' critical value for spins at the lattice spacing, and proliferation leads to other types of couplings as well [10]. In other words, the critical temperature for the *coarse-grained* spin variables is now different from, say, the Onsager value, in the case of the Ising spin system; less or more, depending upon the sign of the fixed point values of the next nearest neighbour (NNN), four-spin, etc couplings generated [10, 11]. We are assured, however, that at the fixed point there will only be two relevant scaling fields (linear combinations, respectively, of the odd and even couplings), one like an external field and the other temperature like [10].

In the discussion that follows, we present an Ornstein–Zernike-like expansion [12] for the matrix elements of the FST at a generic length scale, where in place of the direct correlation function there appear the correlation functions computed over the NN-bonds connecting a site to occupied neighbours. In this computation, we will take the bare value of the critical NN coupling, which can be justified in the following manner:

(i) When the NNN and higher-order couplings at the fixed point of the RG transformation are positive, neglecting them would mean that NN correlations would have to be corrected upwards at the critical temperature. Evaluating the NN bonds at T_c^{bare} does precisely this, since in this case $T_c^{\text{bare}} < T_c^{\text{renorml}}$.

(ii) When the higher order couplings at the fixed point are negative, neglecting them would mean that the NN correlations would have to be corrected downwards at the critical temperature. Now $T_c^{\text{renorml}} < T_c^{\text{bare}}$ and using the bare critical value of the NN bonds again has an effect in the right direction on the NN correlations. From this, it is clear how the general case can be argued.

In essence, therefore, our scheme of restricting the couplings to the bare ones but renormalizing the NN correlations by summing over different connecting paths seems

to effectively take into account some features of the proliferation one would expect when dealing with coarse-grained variables.

We now present an explicit calculation along the lines of the open-closed treatment of the boundary conditions as discussed in [2].

2.1. 'Open' boundary conditions

We start by considering intersections of type 1 and 2 bordered by 'empty' sites. ('Open' boundary conditions, cf [1, 2].)

In figure 2 we show an intersection of type 1 *belonging to an infinite connected cluster.* This connectedness property is ensured by the transverse 'backbone' of occupied sites [3]). We would like to compute the probability $M_{1,2}$ that the grey site is occupied. Notice that the grey site is neighboured by two empty sites and an occupied one. We first ask the question whether its fourth neighbour belongs to the cluster or not. Then, according to the outcome of this question, we compute the probability that the grey site itself belongs to the cluster. This 'flow diagram' is illustrated in figure 2, and yields

$$M_{1,2}^{\text{open}} = p_1 p_2 + (1 - p_1) p_1 . \tag{1}$$

Here we take p_k to be the conditional probability for a site to be 'up', given that k of its neighbours are 'up', or belong to the connected cluster. Note that in figure 2(c), for example, the absence of the bond between sites 1 and 2 has led to the formation of a *vacancy* at site 2. The subsequent probability for the occupation of the grey site is to be computed given one occupied (up) neighbour, and *not* two neighbours, one of which is up and the other down.

We make a rough estimate of p_k neglecting all but the NN correlations,

$$p_j = \frac{e^{kK_c}}{e^{kK_c} + e^{-kK_c}} \tag{2}$$

where the Ising Hamiltonian is $-\mathcal{H}/k_B T = K \sum_{(ij)} \sigma_i \sigma_j$, $\sigma_i = \pm 1$, and the Onsager critical point is at $K_c = \frac{1}{2} \ln(\sqrt{2} + 1)$.

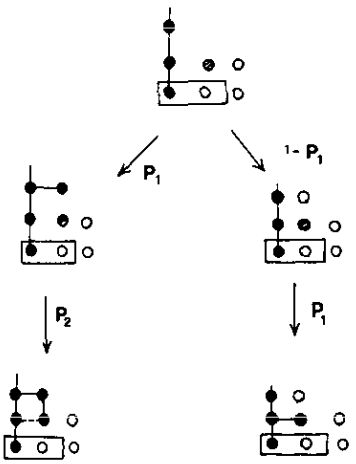

Figure 2. The 'flow diagram' for the computation of the matrix element $M_{1,2}^{\text{open}}$ (see text). Successive bonds over the connected paths leading to the growth site (grey) are shown as full lines. Broken lines from the bond end to its occupied NN indicate interactions that contribute to the computation of the bond probability.

From figure 3 we can analogously write down the probability that a type 2 configuration is followed in the transverse direction again by a type 2 configuration, namely,

$$M_{2,2}^{\text{open}} = p_1 p_3 + (1-p_1)\, p_2. \tag{3}$$

From (1)–(3) we get

$$M_{1,2} = 0.810\,65 \qquad M_{2,2} = 0.910\,18. \tag{4}$$

By definition, $M_{1,1} = 1 - M_{1,2}$, $M_{2,1} = 1 - M_{2,2}$ and the fixed point of the transformation

$$C_i' = \sum_j M_{ji} C_j \tag{5}$$

is given by

$$C_1^* = \left(1 + \frac{M_{1,2}}{M_{2,1}}\right)^{-1}. \tag{6}$$

The fractal dimension of the cluster is

$$D = 1 + \frac{\ln(2 - C_1)}{\ln 2}. \tag{7}$$

Using our lowest order results (4), we find

$$D_{\text{open}} = 1.9262 \tag{8}$$

for open boundary conditions only [2].

2.2. *'Closed' boundary conditions*

We could also compute the matrix M for the case that the segment of the interesecting strip we consider is bordered on the right by sites that also belong to the cluster (are occupied). The flow diagrams for configurations of type 1 and 2 are shown in figures 4 and 5. We find

$$\begin{aligned} M_{1,2}^{\text{closed}} &= p_1\{p_1[p_2 p_3 + (1-p_2)\, p_2] + (1-p_1)[p_1 p_3 + (1-p_1)\, p_2]\} + (1-p_1) M_{1,2}^{\text{open}} \\ M_{2,2}^{\text{closed}} &= p_1\{p_1 p_2 p_4 + p_1(1-p_2)\, p_3 + (1-p_1)\, p_1 p_4 + (1-p_1)^2 p_3\} + (1-p_1) M_{2,2}^{\text{open}}. \end{aligned} \tag{9}$$

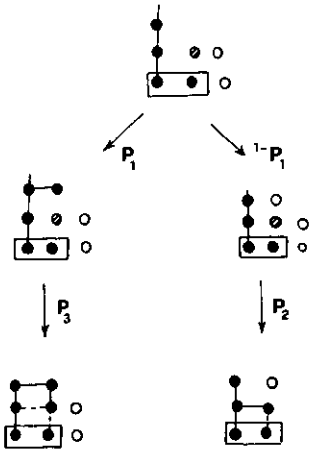

Figure 3. Diagram for the computation of the matrix element $M_{2,2}^{\text{open}}$.

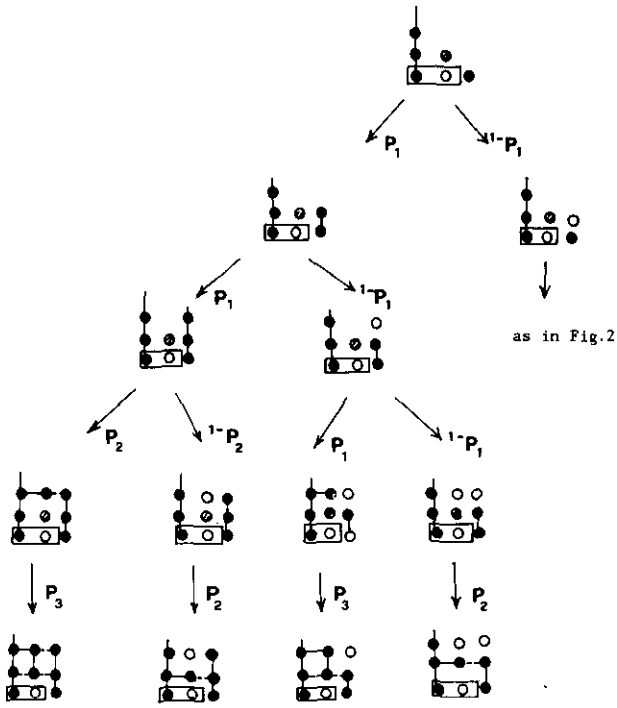

Figure 4. Diagram for the computation of the matrix element $M^{\text{closed}}_{1,2}$.

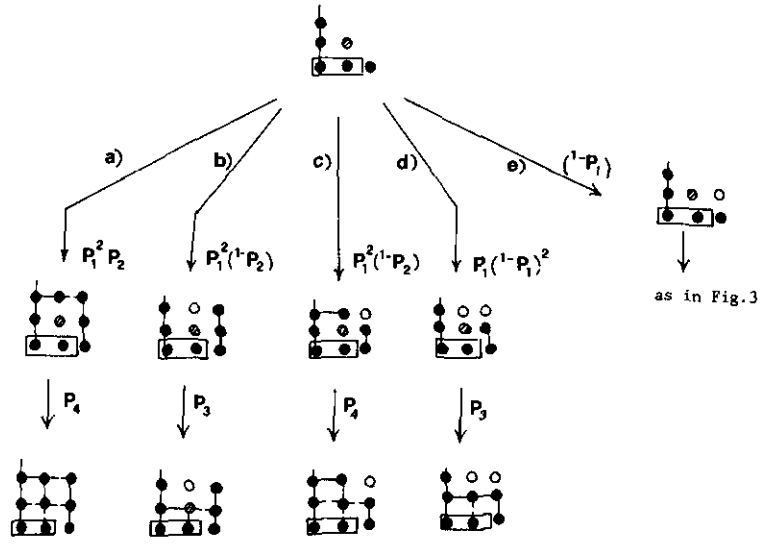

Figure 5. Diagram for the computation of the matrix element $M^{\text{closed}}_{2,2}$. The branchings over the paths (a)–(d) are identical with those in figure 4.

With p_k as given in equation (2), we have

$$M_{1,2}^{\text{closed}} = 0.8869 \qquad M_{2,2}^{\text{closed}} = 0.9484. \tag{10}$$

The resulting fractal dimension is

$$D_{\text{closed}} = 1.959 \tag{11}$$

for closed boundary conditions only [2].

It has previously been discussed in great detail [1, 2, 13] that one may compute the probabilities that segments on the intersection set are neighboured by vacant or occupied cells in terms of the probabilities C_1, C_2. This enables us to compute the properly weighted matrix elements within the so-called open-closed approximation [2] and a new fixed point for C_1. With [1, 2]

$$P_{\text{closed}} = \frac{C_2}{1+\frac{1}{2}C_1} \tag{12}$$

the solution for the fixed point is

$$C_1^* = \frac{1}{2A}\{M_{1,2}^{\text{closed}} + 2M_{2,1}^{\text{closed}} - \tfrac{3}{2}M_{2,1}^{\text{open}} - [(\tfrac{3}{2}M_{2,1}^{\text{open}} - M_{1,2}^{\text{closed}} - 2M_{2,1}^{\text{closed}})^2 - 4M_{2,1}^{\text{closed}}A]^{1/2}\} \tag{13}$$

where

$$A = M_{1,2}^{\text{closed}} + M_{2,1}^{\text{closed}} - \tfrac{3}{2}(M_{1,2}^{\text{open}} + M_{2,1}^{\text{open}}).$$

Using (10), we find in the self-consistent 'open-closed' approximation,

$$D = 1.957. \tag{14}$$

To make sure that our successive approximations are not going astray we have also computed larger diagrams where the occupacy of not only the NN but also the NNN in the vertical direction of our chosen site was checked, before its probability of belonging to the cluster was finally computed. The flow diagram for the open boundary case conditional to a configuration of type 1, is illustrated in figure 6.

The matrix elements are

$$\begin{aligned}
M_{1,2}^{\text{open}}(\text{II}) &= (1-p_1)M_{1,2}^{\text{open}}(\text{I}) + p_1[(1-p_2)p_1 + p_2^2] \\
M_{2,2}^{\text{open}}(\text{II}) &= (1-p_1)M_{2,2}^{\text{open}}(\text{I}) + p_1[p_2p_3 + (1-p_2)p_2]. \\
M_{j,2}^{\text{closed}}(\text{II}) &= (1-p_1)M_{j,2}^{\text{open}}(\text{II}) + p_1\{p_1[p_1[p_2\tilde{C}_j + (1-p_2)\tilde{B}_j] + (1-p_1) \\
&\quad \times [p_1\tilde{C}_j + (1-p_1)\tilde{B}_j]] + (1-p_1)[p_1\tilde{B}_j + (1-p_1)\tilde{A}_j]\} \qquad j = 1, 2
\end{aligned} \tag{15}$$

where I and II denote the lowest- and higher-order computations respectively, and

$$\begin{aligned}
\tilde{C}_j &= p_3p_{2+j} + (1-p_3)p_{1+j} \\
\tilde{A}_j &= p_1p_{2+j} + (1-p_1)p_{1+j} \\
\tilde{B}_j &= p_2p_{2+j} + (1-p_2)p_{1+j}.
\end{aligned} \tag{16}$$

The fractal dimension for the open and closed boundary conditions are

$$D_{\text{II}}^{\text{open}} = 1.9337 \qquad D_{\text{II}}^{\text{closed}} = 1.9634 \tag{17}$$

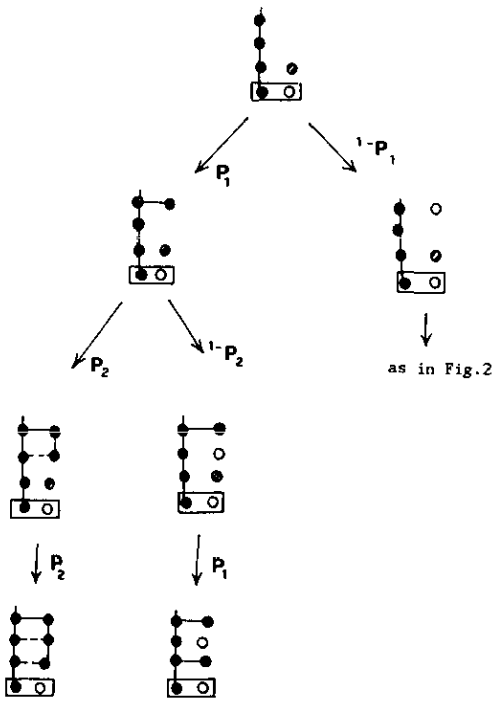

Figure 6. Diagram for the computation of higher order corrections to $M_{1,2}^{open}$.

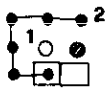

Figure 7. Examples of a class of diagrams that should also be included for a systematic expansion of the matrix elements.

which nicely bracket the desired value. The open-closed self-consistent approximation to this order is given, from (2), (13), (15) and (16), to be

$$D_{II} = 1.961. \tag{18}$$

It should be noted that in this order the device of ensuring connectedness in the cluster by inserting a straight backbone tends to overestimate the matrix elements $M_{i,2}$. As illustrated in figure 7, at this order there are alternative paths by which site '2' neighbouring the grey site could be connected to the cluster, without site '1' being occupied. Thus our 'higher order' computation with a flow diagram as in figure 6 is a 'worst case' scenario for how big the contribution from successive orders could be.

3. Generalization to the Potts case

The generalization to the q-state Potts model, now with connected clusters of identical spins, is straightforward. The Potts Hamiltonian is

$$-H/k_B T = qK \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$$

where δ denotes the Kroneker delta and the critical coupling is given by $\exp(qK_c) = 1+\sqrt{q}$. At the critical point, the conditional probability for a spin with k NNs that are members of the connected cluster, to also belong to the cluster is

$$p_k = \frac{e^{qkK_c}}{e^{qkK_c} + q - 1} \tag{19}$$

to the same approximation as in (2). The flow diagrams for the evaluation of the matrix elements, and, eventually, the expression for the matrix elements in terms of p_k are identical to the calculation for the Ising case, so we just report our results in table 1.

Table 1. Fractal dimension D of the critical Ising, Potts clusters. Here x_H is the magnetic exponent at the tricritical point of the dilute $(q-1)$ state Potts model.

FST results	$q=2$ (Ising)	$q=3$	$q=4$
Open (I)	1.9262	1.8736	1.8328
Closed (I)	1.9597	1.9258	1.8969
Open-closed (I)	1.9570	1.9181	1.8841
Open-closed (II)	1.9612		
$2-x_H$	1.9479	1.9250	1.9047

The exact value for the fractal dimension of Ising clusters in two dimensions has been argued by Stella and Vandezarde [5] to be $D = 2 - x_H$, where x_H is the magnetic exponent associated with the tricritical point of the $q=1$ Potts model, which happens to fall into the same class as the Ising critical point under the conformal classification scheme [14]. Duplantier and Saleur [7] use a direct mapping of the problem onto the O(n) model (with $n=1$) at its critical point, from which they are able to extract the same x_H. They moreover point out [7, 15] that there is a geometrical equivalence between the low-temperature phase of the O(n) model and the $q=n^2$ critical Potts model, as well as between the critical O(n) model and the *tricritical* point of the $q=n^2$ Potts model.

Although the equivalence between the critical cluster problem and the O(n) model holds only for $n=1$, the fractal dimensions computed from the FST are numerically very close to $2-x_H$, where x_H is the magnetic exponent for the tricritical point of the dilute $q=2$ and the $q=3$ Potts models, as can be seen from table 1.

4. Ising and Potts 'droplets'

It has been pointed out by Coniglio *et al* [8, 9] that the fractal dimension of Potts 'droplets' are given in $d=2$, by

$$D_{\text{droplet}} = 2 - x'_H$$

where x'_H is now the magnetic exponent at the critical point of the q-state Potts model (or the 'dense' state of the O(n) model with $n=\sqrt{q}$, by virtue of their geometrical equivalence [7, 15]).

'Droplets' as opposed to 'clusters' are made up of NN sites that have an expectation of being in the same state over and above that coming from pure chance. Since in a

q-state model any two spins have a $1/q$ probability of being in the same state by chance, it is customary to define the 'subtracted' NN correlation function

$$\tilde{p} = \frac{q\langle \delta_{\sigma_i \sigma_j} \rangle - 1}{q-1} \tag{20}$$

where $\langle \ldots \rangle$ is the thermal expectation value. Likewise we may define subtracted conditional probabilities $\tilde{p}_k$,

$$\tilde{p}_k = \frac{qp_k - 1}{q-1} \tag{21}$$

that a spin is in the same state as its k identical nearest neighbours, over and above the probability that it is so by pure chance. From (19) we see that

$$\tilde{p}_k = (1 - e^{-qkK_c}) p_k. \tag{22}$$

We have repeated the same procedure as outlined in section 2, with $\tilde{p}_k$ substituted for the p_k. We report our results in table 2, together with the known exact values for the fractal dimension. As one considers higher order contributions to the FST [1, 2, 3], improvement is slower than in the cluster case. Nevertheless, comparison with exact results seems encouraging.

Table 2. Fractal dimension of the critical Ising and Potts droplets. Here x'_H is the magnetic exponent at the critical point of the q-state Potts model.

FST results	$q=2$ (Ising)	$q=3$	$q=4$
Open (I)	1.7678	1.7297	1.7004
Closed (I)	1.8396	1.8095	1.7852
Open-closed (I)	1.8153	1.7817	1.7525
Open (II)	1.7806		
Closed (II)	1.8585		
Open-closed (II)	1.8414		
$2-x'_H$	$\frac{15}{8}=1.875$	$\frac{28}{15}=1.8666$	$\frac{15}{8}=1.875$

5. Discussion

It is worthwhile pointing out a number of connections and parallels between the present method and the renormalization group [10, 17], as well as a number of more traditional approaches.

Let us denote by $\{n\}$ the lattice gas variables describing the Ising or Potts clusters in question. One may regard the renormalization group transformation as a way of determining the equilibrium distribution governing the $\{n\}$ at criticality, using the requirements of scale invariance, once the universality class has been correctly identified. The correlation function $g(r) = \langle n_i n_{i+r} \rangle$ has the following asymptotic scaling behaviour as $1/r \to 0$ at criticality,

$$g'(r') \sim l^{-2(d-y_H)} g(r/l) \tag{23}$$

where the primed quantities refer to the system scaled by a factor l. The so-called magnetic scaling index y_H is given by $y_H = \ln \lambda_H / \ln l$, when λ_H is the eigenvalue greater than 1 associated with the odd subspace of the renormalization group transformation, evaluated at the appropriate fixed point. Then $y_H = D$, the fractal dimension of the clusters [18, 5, 8, 9]. Note that $y_H + x_H = d$. For a given coarse-graining process, one may derive [10, 19] a relation between λ_H and the ϕ_a^H, components of the associated eigenvector in the odd subspace of the RG transformation. From the RG equation

$$e^{G+\mathcal{H}'(n')} = \sum_{\{n\}} P(n', n)\, e^{\mathcal{H}(n)}$$

where G is the cell contribution to the free energy, and $P(n', n)$ specifies the coarse-graining procedure, with $\Sigma_{\{n'\}} P(n', n) = 1$, one has the identity

$$g'(r'_{i'j'}) = \sum_{\{n'\}} n'_{i'} n'_{j'} \langle P(n', n) \rangle \tag{24}$$

where the expectation value is taken with respect to the unprimed Hamiltonian. Let $P(n', n)$ be factorizable, as is usually the case, over the coarse-graining cells

$$P(n', n) = \prod_{i'} P_{i'}(n', n)$$

and expand $P_{i'}(n', n)$ over products of the primed and unprimed variables, such as single spin, NN pair etc,

$$P_{i'}(n', n) = \sum_{a,b} v_{ab} n_a n'_b$$

where b runs only over the null set and single spin and v_{ab} are constant coefficients. Then using (23) and (24) we find [10]

$$\phi_s^H = l^{-(d-y_H)} \sum_{\text{odd}} \tilde{v}_{as} \phi_a^H \tag{25}$$

where s stands for 'single spin' and $\tilde{v}_{as}$ are simply related to v_{as}. It should be recalled that the matrix elements—and thus also the eigenvectors—of the real space RG transformation are determined entirely in terms of expectation values [10, 17] of products of spins within a selected cell or adjacent cells, as a function of the interactions contained within this group of cells. The expectation values computed are conditional on the values of the transformed (primed) spins. Thus (25) relates D directly to the short-range conditional correlation functions entering into the RG transformation.

Let us now consider the scaling relation

$$M(r) \sim l^{D-1} M(r/l)$$

where

$$M(r) = \int_{x \in I(r)} n(x) \tag{26}$$

is the total 'mass' within a region of size r on the intersection set I of the cluster with a $(d-1)$-dimensional manifold. Asymptotically, (as $1/r \to 0$), this leads [1, 2], for $d = 2$ and $l = 2$, to the following relationship for D in terms of the conditional expectation value over a cell at a generic scale, namely

$$\langle n_1 + n_2 \rangle_{n'=1} = 2^{D-1} \tag{27}$$

where n' is the coarse-grained cell variable. The brackets denote an average over a yet undetermined distribution governing $\{n\}_I$. The requirement that this distribution should be invariant under a translation in the transverse direction generates the FST. Finally, in close analogy to (25), equation (27) can be re-expressed as a relationship between the scaling index D and the elements of the eigenvector of the FST with unit eigenvalue [1, 2]

$$C_1 + 2C_2 = 2^{D-1} \tag{28}$$

(see equation (7)).

In the non-interacting case, where all the expectation values can be computed exactly, we have checked that (24) and (27) lead to the same expressions, with the choice

$$P_{i'}(n'_{i'}, n) = 1 - (n'_{n'} + \rho_{i'}) + 2n'_{i'}\rho_{i'}$$

$$\rho_{i'} = n_1(i') + n_2(i') - 2n_1(i')n_2(i')$$

where $n_k(i')$ refers to the kth site within the i' cell.

We now turn our attention to the construction of the FST, which, for equilibrium models as in the present computation, follows formally along the lines of Glauber's [20] approach to the time-dependent statistics of the Ising model, with time now being replaced by y, the spatial dimension transverse to the intersection set.

Let $\Omega_m(S, y)$ be the probability of a given configuration S of n on the intersection set, at a length scale 2^m. We assume that the Ω do not depend on m at criticality, at least for sufficiently large m. Now we will consider a subclass of processes by which S can change under a translation in the y direction. In a cell of size 2^{m+1} that belongs to the set, by definition there is at least one site that is occupied. Define $w_i^{(m)}(1, n_2)$ to be the rate per unit translation at which n_2, a site in cell i that is neighboured by an occupied site, changes its state (i.e. $n_2 \to (n_2+1) \bmod_2$). We assume $w_i^{(m)}(1, n_2) = w_i^{(m)}(n_1, 1)$, so we will just concentrate on the first process. Moreover, the $w_i^{(m)}$ are assumed to be independent of m. In general the w_i may depend upon the values of all the n. The fact that we keep n_1 fixed restricts us to a set of configurations where connectivity of the cluster is ensured in the transverse direction. We may now write down a 'master equation' for the desired probability distribution:

$$\frac{\mathrm{d}}{\mathrm{d}y}\Omega(S, y) = -\sum_i w_i(1, n_2)\Omega(S, y) + \sum_i w_i(1, (n_2+1) \bmod_2)\Omega(S', y) \tag{29}$$

where S' differs from S only in the second site of the ith cell.

The 'transition rates' may be chosen to have the following form,

$$w(1, n_2) = n_2 T_{21} + (1 - n_2) T_{12} \tag{30}$$

where the T_{ij} may depend upon the values of all other n than n_2. By standard procedures we next obtain for the jth cell, in obvious notation

$$\frac{\mathrm{d}}{\mathrm{d}y}\langle 1 \cdot n_2(j)\rangle = -\langle 1 \cdot n_2(j)\rangle\langle T_{21}\rangle + \langle T_{12}\rangle - \langle 1 \cdot n_2(j)\rangle\langle T_{12}\rangle \tag{31}$$

where the expectation values are taken over $\Omega(S, y)$. We have used the fact that $n^2 = n$, $n(n+1) \bmod_2 = 0$ and that

$$\sum_S n_2(j) T_{12}\Omega(S') = \sum_S (1 - n_2(j)) T_{12}\Omega(S)$$

since S' differs from S only in that $n_2(j) \to (n_2(j)+1) \bmod_2$. Moreover, by construction the T_{ij} do not depend upon n.

In the translationally invariant state we have, then

$$\langle 1 \cdot n_2 \rangle = \frac{\langle T_{12} \rangle}{\langle T_{21} \rangle + \langle T_{12} \rangle}. \tag{32}$$

This is identical to the fixed point equation (6), if we notice that $C_2 = \langle 1 \cdot n_2 \rangle$. We now carry the analogy one step further by examining the T_{ij}.

In equilibrium, the w are expected to satisfy

$$\frac{w_i(1, n_2)}{w_i(1, n_2+1) \bmod_2} = \frac{p_i(1, (n_2+1) \bmod_2)}{p_i(1, n_2)}$$

where $p_i(1, n)$ is the probability that the lattice gas variable at site 2 of the ith cell has the value n, given that site 1 is occupied, as a function of the surrounding configuration. Due to the form (30) the T_{ij} satisfy

$$\frac{T_{12}}{T_{21}} = \frac{p(1, (n+1) \bmod_2)}{p(1, n)}. \tag{33}$$

The $p(1, n)$ are proportional to the Boltzmann factors $\exp(-H_1/kT)$, where H_1 now involves interactions not only between sites 1 and 2 inside the cell on the intersection set but also along paths that do not lie along the chain. Our strategy is to expand them in a sum over linear paths connecting the sites 1 and 2, taking a product of the Boltzmann factors along each NN bond, computed at the critical temperature of the underlying Ising spin system. In spirit, this is similar to the Ornstein-Zernike expansion for the correlation function [12], where one sums over paths of all lengths, taking products of so-called 'direct' (or short-ranged) correlation functions along the paths, with a configurational average being then taken over the intermediate vertices.

Finally, it should be noticed that (see figures 2–6) we have imposed a further constraint on the configuration sums appearing in (31) or (32) by requiring that the connectivity in the transverse direction be maintained in the second, third, ..., steps. These configuration averages also include the properly weighted contributions from 'open' and 'closed' boundary conditions.

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